

Differential Meadows

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Abstract

A meadow is a zero totalised field ($0^{-1} = 0$), and a cancellation meadow is a meadow without proper zero divisors. In this paper we consider differential meadows, i.e., meadows equipped with differentiation operators. We give an equational axiomatization of these operators and thus obtain a finite basis for differential cancellation meadows. Using the Zariski topology we prove the existence of a differential cancellation meadow.

1 Introduction

A *meadow* is an algebra in the signature of fields with an inverse operator that satisfies the equations of commutative rings with unit (*CRU*) together with

$$\begin{aligned} (x^{-1})^{-1} &= x & (Ref) \\ x \cdot x \cdot x^{-1} &= x & (RIL) \end{aligned}$$

where the names of the equations abbreviate *Reflection* and *Restricted Inverse Law*, respectively. Meadows were introduced in [2].

In [1] it was shown that the variety of meadows satisfies precisely those equations which are valid in all so-called zero totalised fields (*ZTFs*). A *ZTF* is a field equipped with an inverse operator $(_)^{-1}$ that has been made total by putting $0^{-1} = 0$. Alternatively and following [4], we will qualify a zero totalised field as a *cancellation meadow* if it enjoys the following cancellation property:

$$x \neq 0 \wedge x \cdot y = x \cdot z \implies y = z. \quad (1)$$

The mentioned result from [1] may be viewed as a completeness theorem: *CRU* + *Ref* + *RIL* completely axiomatises the equational theory $E(ZTF)$ of the class *ZTF* of zero totalised fields. Another way of looking at this result is that it establishes that $E(ZTF)$ has a finite

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basis. The proof of the finite basis theorem for $E(ZTF)$ in [1] makes use of the existence of maximal ideals. Although concise and readable, that proof is non-elementary because the existence of maximal ideals requires a non-elementary set theoretic principle, independent of ZF set theory. In [2] a finite basis theorem was established for $E_C(ZTF)$, the closed equations true in ZTF s.

In [3] a proof of the finite basis result for $E(ZTF)$ has been given along the lines of the elementary proof about $E_C(ZTF)$. The proof method is more general than the proof using maximal ideals because it generalizes to extended signatures (see Theorem 1 below). In this paper we apply this result to so-called differential meadows, i.e., meadows equipped with formal variables X_1, \dots, X_n and differential operators

$$\frac{\partial}{\partial X_i}.$$

We provide a short equational axiomatization of the differential operators and thus obtain a finite basis for differential cancellation meadows. This appears to be an elegant axiomatization, e.g., $\frac{\partial}{\partial X_i}(1/x) = -(1/x^2) \cdot \frac{\partial}{\partial X_i}(x)$ follows easily. Finally, we prove the existence of a differential cancellation meadow, using the Zariski topology [7, 6] and a representation result from [3].

The paper is structured as follows: in the next section we recall cancellation meadows and the generic basis theorem, and introduce differential meadows. Then, in Section 3 we prove the existence of a differential cancellation meadow. Some conclusions are drawn in Section 4.

2 Cancellation and Differential Meadows

In this section we fix some notation and explain cancellation meadows and our generic basis theorem in detail. Then we introduce differential meadows.

2.1 Cancellation meadows and a generic basis result

A meadow is an algebra in the signature of fields that satisfies the axioms in Table 1. We write Md for the set of axioms in Table 1, thus (referring to the Introduction) $Md = CRU + Refl + RIL$.

Let IL (Inverse Law) stand for

$$x \neq 0 \implies x \cdot x^{-1} = 1,$$

so IL states that there are no zero divisors. Note that IL and the cancellation property (1) are equivalent. A *cancellation meadow* is a meadow that also satisfies IL .

From the axioms in Md the following identities are derivable:

$$\begin{aligned} (0)^{-1} &= 0, & 0 \cdot x &= 0, \\ (-x)^{-1} &= -(x^{-1}), & x \cdot -y &= -(x \cdot y), \\ (x \cdot y)^{-1} &= x^{-1} \cdot y^{-1}, & -(-x) &= x. \end{aligned}$$

$$\begin{aligned}
(x + y) + z &= x + (y + z) \\
x + y &= y + x \\
x + 0 &= x \\
x + (-x) &= 0 \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z) \\
x \cdot y &= y \cdot x \\
1 \cdot x &= x \\
x \cdot (y + z) &= x \cdot y + x \cdot z \\
(x^{-1})^{-1} &= x \\
x \cdot (x \cdot x^{-1}) &= x
\end{aligned}$$

Table 1: The set Md of axioms for meadows

We write $\Sigma_m = (0, 1, +, \cdot, -, ^{-1})$ for the signature of (cancellation) meadows. Furthermore, we often write $1/t$ or $\frac{1}{t}$ for t^{-1} , tu for $t \cdot u$, t/u for $t \cdot (1/u)$, $t - u$ for $t + (-u)$, and freely use numerals and exponentiation with constant integer exponents. We further use the notation

$$1_x \text{ for } \frac{x}{x} \quad \text{and} \quad 0_x \text{ for } 1 - 1_x.$$

Note that for all terms t , $(1_t)^2 = 1_t$, $1_t \cdot 0_t = 0$ and $(0_t)^2 = 0_t$. We call an expression 1_t a *pseudo unit* because it is almost equivalent to the unit 1, and for a similar reason we say that 0_t is a *pseudo zero*.

The basis result from [3] admits generalization if pseudo units and pseudo zeros propagate in the context rule for equational logic. We recall the precise definition of this form of propagation from that paper.

Definition 1. Let Σ be an extension of $\Sigma_m = (0, 1, +, \cdot, -, ^{-1})$, the signature of meadows, and let $E \supseteq Md$ be a set of equations over Σ . Then

1. (Σ, E) has the **propagation property for pseudo units** if for each pair of Σ -terms t, r and context $C[\]$,

$$E \vdash 1_t \cdot C[r] = 1_t \cdot C[1_t \cdot r].$$

2. (Σ, E) has the **propagation property for pseudo zeros** if for each pair of Σ -terms t, r and context $C[\]$,

$$E \vdash 0_t \cdot C[r] = 0_t \cdot C[0_t \cdot r].$$

We now recall our generic basis result from [3]:

Theorem 1 (Generic basis theorem for cancellation meadows). *If $\Sigma \supseteq \Sigma_m$, $E \supseteq Md$ is a set of equations over Σ , and (Σ, E) has the pseudo unit propagation property and the pseudo zero propagation property, then E is a basis (a complete axiomatisation) of $\text{Mod}_\Sigma(E \cup IL)$.*

$$\frac{\partial}{\partial X_i}(x + y) = \frac{\partial}{\partial X_i}(x) + \frac{\partial}{\partial X_i}(y) \quad (\text{D1})$$

$$\frac{\partial}{\partial X_i}(x \cdot y) = \frac{\partial}{\partial X_i}(x) \cdot y + x \cdot \frac{\partial}{\partial X_i}(y) \quad (\text{D2})$$

$$\frac{\partial}{\partial X_i}(x \cdot x^{-1}) = 0 \quad (\text{D3})$$

$$\frac{\partial}{\partial X_i}(X_i) = 1 \quad (\text{D4})$$

$$\frac{\partial}{\partial X_i}(X_j) = 0 \quad \text{if } i \neq j \quad (\text{D5})$$

Table 2: The set of axioms DE

2.2 Differential Meadows

Given some $n \geq 1$ we extend the signature Σ_m of meadows with differentiation operators and constants X_1, \dots, X_n to model functions to be differentiated:

$$\frac{\partial}{\partial X_i} : \mathbb{M} \rightarrow \mathbb{M}$$

for $i = 1, \dots, n$ and some meadow \mathbb{M} . We write Σ_{md} for this extended signature. Equational axioms for $\frac{\partial}{\partial X_i}$ are given in Table 2, where D4 and D5 define n^2 equational axioms. Observe that the Md axioms together with D3 imply $\frac{\partial}{\partial X_i}(0) = 0$. Furthermore, using axiom D1 one easily proves: $\frac{\partial}{\partial X_i}(-x) = -\frac{\partial}{\partial X_i}(x)$.

First we establish the expected corollary of Theorem 1:

Corollary 1. *The set of axioms $Md \cup DE$ (see Tables 1 and 2) is a finite basis (a complete axiomatisation) of $Mod_{\Sigma_{md}}(Md \cup DE \cup IL)$.*

Proof. The pseudo unit propagation property requires a check for $\frac{\partial}{\partial X_i}(\cdot)$ only:

$$\frac{\partial}{\partial X_i}(1_t \cdot r) = \frac{\partial}{\partial X_i}(1_t) \cdot r + 1_t \cdot \frac{\partial}{\partial X_i}(r) = 1_t \cdot \frac{\partial}{\partial X_i}(r). \quad (2)$$

Multiplication with 1_t now yields the property. From (2) we get

$$0_t \cdot \frac{\partial}{\partial X_i}(r) = \frac{\partial}{\partial X_i}(r) - 1_t \cdot \frac{\partial}{\partial X_i}(r) \stackrel{(2)}{=} \frac{\partial}{\partial X_i}(r) - \frac{\partial}{\partial X_i}(1_t \cdot r) = \frac{\partial}{\partial X_i}(0_t \cdot r)$$

and multiplication with 0_t then yields the pseudo zero propagation property. \square

A *differential meadow* is a meadow equipped with formal variables X_1, \dots, X_n and differentiation operators $\frac{\partial}{\partial X_i}(\cdot)$ that satisfies the axioms in DE .

We conclude this section with an elegant consequence of the fact that we are working in the setting of meadows, namely the consequence that the differential of an inverse follows from the *DE* axioms.

Proposition 1. $Md \cup DE \vdash \frac{\partial}{\partial X_i}(1/x) = -(1/x^2) \cdot \frac{\partial}{\partial X_i}(x).$

Proof. By axioms D3 and D2, $0 = \frac{\partial}{\partial X_i}(x/x) = \frac{\partial}{\partial X_i}(x) \cdot 1/x + x \cdot \frac{\partial}{\partial X_i}(1/x)$, so

$$\begin{aligned} 0 &= 0 \cdot (1/x) = \frac{\partial}{\partial X_i}(x/x) \cdot (1/x) = \frac{\partial}{\partial X_i}(x) \cdot 1/x^2 + (x/x) \cdot \frac{\partial}{\partial X_i}(1/x) \\ &\stackrel{(2)}{=} 1/x^2 \cdot \frac{\partial}{\partial X_i}(x) + \frac{\partial}{\partial X_i}((x/x) \cdot (1/x)) \stackrel{RIL}{=} 1/x^2 \cdot \frac{\partial}{\partial X_i}(x) + \frac{\partial}{\partial X_i}(1/x), \end{aligned}$$

and hence

$$\frac{\partial}{\partial X_i}(1/x) = -(1/x^2) \cdot \frac{\partial}{\partial X_i}(x).$$

□

3 Existence of Differential Meadows

In this section we show the *existence* of differential meadows with formal variables X_1, \dots, X_n for arbitrary finite $n > 0$. First we define a particular cancellation meadow, and then we expand this meadow to a differential cancellation meadow by adding formal differentiation.

3.1 The Zariski topology congruence over \mathbb{C}_0^n

We will use some terminology from algebraic geometry, in particular we will use the Zariski topology [7, 6]. Open (closed) sets in this topology will be indicated as Z-open (Z-closed). Recall that complements of Z-closed sets are Z-open and complements of Z-open sets are Z-closed, finite unions of Z-closed sets are Z-closed, and intersections of Z-closed sets are Z-closed. Let \mathbb{C}_0 denote the zero-totalized expansion of the complex numbers. We will make use of the following facts:

1. The solutions of a set of polynomial equations (with n or less variables) within \mathbb{C}_0^n constitute a Z-closed subset of \mathbb{C}_0^n . Here 'polynomial' has the conventional meaning, not involving division. Taking equations $1 = 0$ and $0 = 0$ respectively, it follows that both \emptyset and \mathbb{C}_0^n are Z-closed (and Z-open as well).
2. Intersections of non-empty Z-open sets are non-empty.

In the following we consider terms

$$t(\overline{X}) = t(X_1, \dots, X_n)$$

with $t = t(\bar{x})$ a Σ_m -term and we write $T(\Sigma_m(\bar{X}))$ for the set of these terms. For $V \subseteq \mathbb{C}_0^n$ we define the equivalence

$$\equiv_{\mathbb{C}_0^n}^V$$

on $T(\Sigma_m(\bar{X}))$ by $t(\bar{X}) \equiv_{\mathbb{C}_0^n}^V r(\bar{X})$ if each assignment $\bar{X} \mapsto V$ evaluates both sides to equal values in \mathbb{C}_0 . It follows immediately that for each $V \subseteq \mathbb{C}_0^n$, $T(\Sigma_m(\bar{X}))/\equiv_{\mathbb{C}_0^n}^V$ is a meadow. In particular, if $V = \emptyset$ one obtains the trivial meadow ($0 = 1$) as both 0 and 1 satisfy any universal quantification over an empty set. If V is a singleton this quotient is a cancellation meadow. In other cases the meadow may not satisfy the cancellation property. Indeed, suppose that $n = 1$ and $V = \{0, 1\}$ and let $t(X) = X$. Now $t(1) \neq 0$. Thus $t(X) \neq 0$ in $T(\Sigma_m(X))/\equiv_{\mathbb{C}_0}^V$. If that is assumed to be a cancellation meadow, however, one has $1_{t(X)} = 1$, but $1_{t(0)} = 0$, thus refuting $1_{t(X)} = 1$.

We now define the relation \equiv_{ZTC} (Zariski Topology Congruence over \mathbb{C}_0^n) by

$$t \equiv_{ZTC} r \iff \exists V (V \text{ is Z-open, } V \neq \emptyset \text{ and } t \equiv_{\mathbb{C}_0^n}^V r).$$

The relation \equiv_{ZTC} is indeed a congruence for all meadow operators: the equivalence properties follow easily; for $0 \equiv_{ZTC} 0$ and $1 \equiv_{ZTC} 1$, take $V = \mathbb{C}_0^n$, and if $P \equiv_{ZTC} P'$ and $Q \equiv_{ZTC} Q'$, witnessed respectively by V and V' , then $P + P' \equiv_{ZTC} Q + Q'$ and $P \cdot P' \equiv_{ZTC} Q \cdot Q'$ are witnessed by $V \cap V'$ which is Z-open and non-empty because of fact 2 above. Finally $-P \equiv_{ZTC} -P'$ and $(P)^{-1} \equiv_{ZTC} (P')^{-1}$ are both witnessed by V .

In [3] we defined the *Standard Meadow Form* (SMF) representation result for meadow terms. This result implies for $T(\Sigma_m(\bar{X}))/\equiv_{ZTC}$ that each term can be represented by 0 or by p/q with p and q polynomials not equal to 0. We notice that it is decidable whether or not a polynomial equals the 0-polynomial by taking all corresponding products of powers of the X_1, \dots, X_n together and then checking that all coefficients vanish.

A few more words on the SMF representation result. SMFs are defined using levels: an SMF of level 0 is of the form p/q with p and q polynomials, and an SMF of level $k + 1$ is of the form $0_p \cdot P + 1_p \cdot Q$ with P and Q both SMFs of level k . As an example, let P be the SMF of level 1 defined by

$$P = 0_{1-X_1} \cdot \frac{2X_1}{X_2} + 1_{1-X_1} \cdot \frac{1 + X_2 - 2X_1X_3}{8 - X_1X_3^2}.$$

Now in $T(\Sigma_m(\bar{X}))/\equiv_{ZTC}$, the polynomial $1 - X_1$ is on some Z-open non-empty set V not equal to 0 (see fact 1 above), thus $1_{1-X_1} \equiv_{\mathbb{C}_0^n}^V 1$ and $0_{1-X_1} \equiv_{\mathbb{C}_0^n}^V 0$, and hence

$$P \equiv_{ZTC} \frac{1 + X_2 - 2X_1X_3}{8 - X_1X_3^2}.$$

So, in $T(\Sigma_m(\bar{X}))/\equiv_{ZTC}$, the SMF level-hierarchy collapses and terms can be represented by either 0 or by p/q with both p and q polynomials not equal to 0. In the second case $1_{p/q} = 1$ and therefore it is a cancellation meadow. Furthermore, equality is decidable in this model. Indeed to check that $1_p = 1$ (and $0_p = 0$) for a polynomial p it suffices to check that p is not 0 over the complex numbers. Using the SMF representation all closed terms are either 0 or take the form p/q with p and q nonzero polynomials. For q and q' nonzero polynomials we find that $p/q \equiv_{ZTC} p'/q' \iff p \cdot q' - p' \cdot q = 0$ which we have already found to be decidable.

3.2 Constructing a differential cancellation meadow

In $T(\Sigma_m(\overline{X}))/\equiv_{ZTC}$ the differential operators can be defined as follows:

$$\frac{\partial}{\partial X_i}(0) = 0$$

and, using the fact that differentials on polynomials are known,

$$\frac{\partial}{\partial X_i}\left(\frac{p}{q}\right) = \frac{\frac{\partial}{\partial X_i}(p) \cdot q - p \cdot \frac{\partial}{\partial X_i}(q)}{q^2}.$$

Let V be the set of 0-points of q and let $U = \sim V$, the complement of V . Then p/q is differentiable on U and the derivative coincides with the formal derivative used in the definition. This definition is representation independent: consider $p'/q' \equiv_{ZTC} p/q$ with V' the 0-points of q' and $U' = \sim V'$. Then there is some non-empty and Z-open W such that $p/q \equiv_{\mathbb{C}_0^n}^W p'/q'$. Now $W \cap U \cap U'$ is non-empty and Z-open, and on this set,

$$\frac{\partial}{\partial X_i}\left(\frac{p}{q}\right) = \frac{\partial}{\partial X_i}\left(\frac{p'}{q'}\right).$$

So, formal differentiation $\partial/\partial X_i$ preserves the congruence properties. Finally, we check the soundness of the *DE* axioms:

Axiom D1: Consider $t + t'$. In the case that one of t and t' equals 0, axiom D1 is obviously sound. In the remaining case, $t = p/q$ and $t' = p'/q'$ with all polynomials not equal to 0 and $t + t' = \frac{pq' + p'q}{qq'}$. Using ordinary differentiation on polynomials we derive

$$\begin{aligned} \frac{\partial}{\partial X_i}(t + t') &= \frac{\frac{\partial}{\partial X_i}(pq' + p'q) \cdot qq' - (pq' + p'q) \cdot \frac{\partial}{\partial X_i}(qq')}{(qq')^2} \\ &= \frac{\frac{\partial}{\partial X_i}(p) \cdot q \cdot (q')^2 + \frac{\partial}{\partial X_i}(p') \cdot q^2 \cdot q' - p \cdot \frac{\partial}{\partial X_i}(q) \cdot (q')^2 - p' \cdot \frac{\partial}{\partial X_i}(q') \cdot q^2}{(qq')^2} \\ &= \frac{\partial}{\partial X_i}\left(\frac{p}{q}\right) \cdot 1_{(q')^2} + \frac{\partial}{\partial X_i}\left(\frac{p'}{q'}\right) \cdot 1_{q^2} \\ &= \frac{\partial}{\partial X_i}(t) + \frac{\partial}{\partial X_i}(t'). \end{aligned}$$

Axiom D2: Similar.

Axiom D3: Consider t , then either $t = 0$ or $t/t = 1$, and in both cases $\frac{\partial}{\partial X_i}\left(\frac{t}{t}\right) = 0$.

Axioms schemes D4 and D5: We derive

$$\frac{\partial}{\partial X_i}(X_j) = \frac{\partial}{\partial X_i}\left(\frac{X_j}{1}\right) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, by adding formal differentiation to $T(\Sigma_m(\overline{X}))$ we constructed a differential cancellation meadow.

4 Conclusions

In this paper we introduced differential meadows. We provided a finite equational basis for differential cancellation meadows and proved their existence by a construction based on the Zariski topology.

Differential meadows generalize differential fields in the same way as meadows generalize fields. As stated in [1], exactly the von Neumann regular rings admit expansion to a meadow. The general question, however, which meadows can be expanded to differential meadows that satisfy the *DE* axioms is left open. In [5] finite meadows have been characterized as direct sums of finite fields. The existence of differential meadows over a finite meadow is in particular left for further analysis.

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